§ Appendix B: Complex Numbers

<u>Motivation</u>: It would be nice if an $n \times n$ matrix A had n eigenvalues (counting multiplicity). For this to happen we'd need the characteristic polynomial $\det(A - \lambda I)$ to have n roots (counting multiplicity).

However, this doesn't happen in general for polynomials to have degree-many roots over the real numbers. However, this does hold over the complex numbers $\mathbb{C}$.

Recall we have $i = \sqrt{-1}$ and can form the complex numbers as the set of all

$$z = a + bi$$

for real numbers a, b. Here the <u>real part</u> of z, <u>$\operatorname{Re} z$</u> is a and the <u>imaginary part</u> of z, <u>$\operatorname{Im} z$</u> is b.

Operations in $\mathbb{C}$

1) Addition is the same as in $\mathbb{R}$, $(a+bi)+(c+di) = (a+c)+(b+d)i$

2) Multiplication is the same as in the reals, but need to "foil" often

$$(a+bi)(c+di) = (ac-bd)+(ad+bc)i$$

3) The <u>conjugate</u> of $z = a+bi$ is defined to be $\overline{z} = \overline{a+bi} = a-bi$

4) The <u>modulus</u> of $z = a+bi$ is

$$|z| = \sqrt{z \cdot \overline{z}} = \sqrt{a^2+b^2}$$

Remarks

- Notice $z = \overline{z}$ if and only if z is a real number

- Notice $z \cdot \overline{z}$ is always a real number

- If z is a real number, then $|z|$ is the same thing as the real absolute value.

Properties of $\overline{z}$ and $|z|$

1) $\overline{w+z} = \overline{w} + \overline{z}$

2) $\overline{w \cdot z} = \overline{w} \cdot \overline{z}$

3) $z \cdot \overline{z} = |z|^2 \geq 0$ (only $= 0$ if $z = 0$)

4) $|w \cdot z| = |w| \cdot |z|$

5) $|w + z| \leq |w| + |z|$

6) If $z \neq 0$, then z has a multiplicative inverse

$$\frac{1}{z} = \frac{\overline{z}}{|z|^2}$$

Example

Express $\dfrac{1+3i}{6+8i}$ as a complex number of the form $a + bi$.

Solution

$$\frac{1+3i}{6+8i} = (1+3i) \cdot \frac{1}{6+8i}$$

$$= (1+3i) \cdot \frac{6-8i}{|6+8i|^2}$$

$$= \frac{(1+3i)(6-8i)}{\sqrt{6^2+8^2}}$$

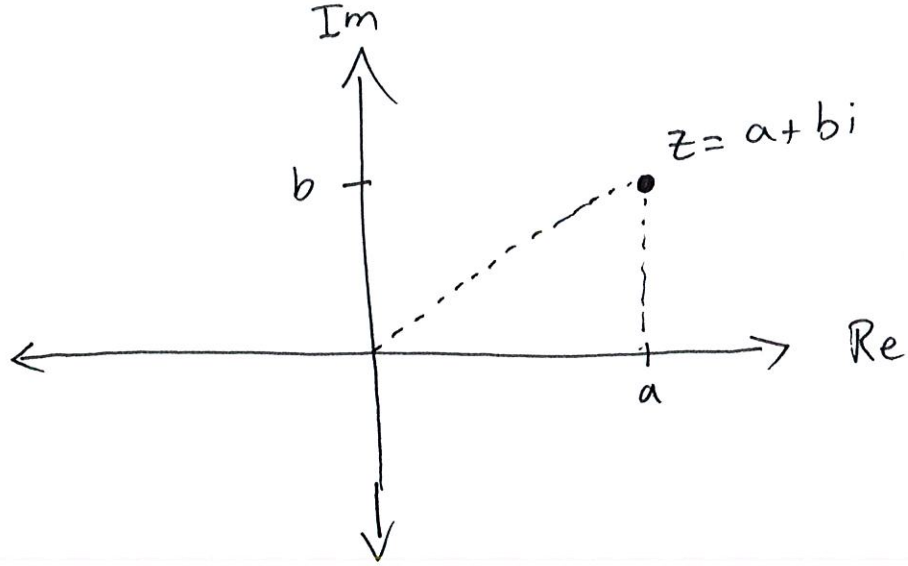
$$= \frac{6-8i+18i+24}{\sqrt{100}}$$

$$= \frac{30+10i}{10}$$

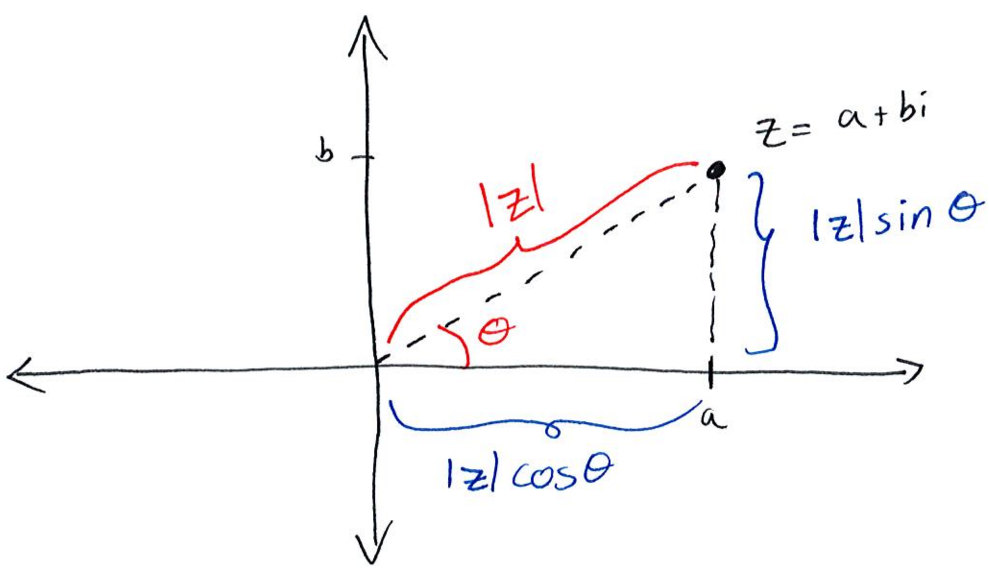
$$= \boxed{3+i}$$

Geometry of $\mathbb{C}$

For a complex number $z = a+bi$ we can associate to it a point in $\mathbb{R}^2$ (a,b). We may view $\mathbb{C}$ as a cartesian plain. with a horizontal real axis and a vertical imaginary axis



Notice this forms a right triangle whose hypotenuse has length $|z|$. We can also express complex numbers in terms of <u>polar coordinates</u>



Here θ is called the <u>argument</u> of z.

Thus if $z = a + bi$, then

$$a = |z| \cos\theta \quad \text{and} \quad bi = i|z| \sin\theta$$

Thus

$$z = a + bi = |z|\cos\theta + i|z|\sin\theta$$

$$\boxed{z = |z|(\cos\theta + i\sin\theta)}$$

is the <u>polar form</u> of z. To convert $z = a+bi$ to polar form you'll need

$$|z| = \sqrt{a^2+b^2} \qquad \theta = \tan^{-1}\left(\frac{b}{a}\right)$$

<u>Theorem</u>

$$e^{i\theta} = \cos\theta + i\sin\theta$$

<u>Proof</u>

Using Maclaurin series:

$$e^{i\theta} = \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!}$$

$$= 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \cdots$$

$$= 1 + i\theta - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{i\theta^5}{5!} \cdots$$

$$= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} \cdots\right)$$

$$= \cos\theta + i\sin\theta$$

Thus we can write complex number z with argument $\arg z = \theta$ as

$$z = |z|\left(\cos(\theta) + i\sin(\theta)\right)$$

$$\boxed{z = |z| e^{i\theta}}$$

Useful for computing powers. For example, writing $r = |z|$, if

$$z = r(\cos\theta + i\sin\theta)$$

then $$z^k = r^k\left(\cos(k\theta) + i\sin(k\theta)\right)$$

Example

Find all solutions to the equation

$$z^{10} = 4^{10}$$

Solution: write $z = r \cdot e^{i\theta}$ where $r = |z|$

$$z^{10} = r^{10} e^{i \cdot 10\theta} = 4^{10} = 4^{10} \cdot e^{i \cdot 0}$$

$$r^{10} (\cos(10\theta) + i\sin(10\theta)) = 4^{10} \cdot (1 + 0i)$$

$$\Rightarrow \boxed{r = 4} \quad \text{and} \quad \begin{cases} \cos(10\theta) = 1 \\ \sin(10\theta) = 0 \end{cases}$$

$$\begin{cases} \cos(10\theta) = 1 \\ \sin(10\theta) = 0 \end{cases} \Rightarrow 10\theta = 0 + 2\pi k \quad k = 1, 2, 3, \ldots$$

$$\Rightarrow \boxed{\theta = \frac{\pi}{5} k \quad k = 0, \ldots, 9 \quad \text{or } k = 1, \ldots, 10}$$

Thus $z = 4 \cdot e^{i \cdot \frac{\pi}{5} k}$ $\quad k = 0, 1, \ldots, 9$

or

$$z = 4 \left(\cos\left(\frac{\pi}{5} k\right) + i \sin\left(\frac{\pi}{5} k\right) \right) \quad k = 0, \ldots, 9$$

Example

If $p(x)$ is a polynomial with real coefficients and z is a complex number with $p(z)=0$, show $p(\bar{z})=0$ as well.

Solution

Write $p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$

for $a_0, \ldots, a_n$ real coefficients.

We have

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0 = 0$$

Now $\overline{}$ both sides

$$\overline{a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0} = \overline{0}$$

zero a real #

$$\overline{a_n z^n} + \overline{a_{n-1} z^{n-1}} + \cdots + \overline{a_1 z} + \overline{a_0} = 0$$

$$\overline{a_n} \cdot \overline{z^n} + \overline{a_{n-1}} \cdot \overline{z^{n-1}} + \cdots + \overline{a_1} \cdot \overline{z} + \overline{a_0} = 0$$

$a_0, \ldots, a_n$ real #s

$$\underbrace{a_n (\bar{z})^n + a_{n-1} \cdot (\bar{z})^{n-1} + \cdots + a_1 (\bar{z}) + a_0}_{p(\bar{z})} = 0$$

$$p(\bar{z}) = 0$$

Example

Identify $z = a+bi$ with vector $\begin{bmatrix} a \\ b \end{bmatrix}$ of $\mathbb{R}^2$. ~~Find~~
In other words $a+bi \equiv \begin{bmatrix} a \\ b \end{bmatrix}$. Find the matrix

$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ with $e^{i\theta}(a+bi) \equiv \left(A \begin{bmatrix} a \\ b \end{bmatrix} \right)^{+}$.

Explain the action of A on $\begin{bmatrix} a \\ b \end{bmatrix}$ geometrically.

Solution

$$
\begin{aligned}
e^{i\theta}(a+bi) &= (\cos\theta + i\sin\theta)(a+bi) \\
&= a\cos\theta + i\cdot a\cdot\sin\theta + i\cdot b\cdot\cos\theta - b\sin\theta \\
&= (a\cos\theta - b\sin\theta) + i(a\sin\theta + b\cos\theta) \\
&\equiv \begin{bmatrix} a\cos\theta - b\sin\theta \\ a\sin\theta + b\cos\theta \end{bmatrix} \\
&= \underbrace{\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}}_{A} \begin{bmatrix} a \\ b \end{bmatrix}
\end{aligned}
$$

A is a rotation matrix so geometrically, the action of A on $\begin{bmatrix} a \\ b \end{bmatrix}$ is rotation counter-clockwise by θ.